ON THE REMAINDERS AND CONVERGENCE OF THE SERIES FOR THE PARTITION FUNCTION†

BY

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1. **Introduction.** The two series under discussion are

$$(1)\quad p(n) = \frac{12^{1/2}}{24n-1}\sum_{k=1}^{N} A_k^*(n)\left(1-\frac{k}{\mu}\right)e^{\mu/k} + R_1(n, N),$$

$$(2)\quad p(n) = \frac{12^{1/2}}{24n-1}\sum_{k=1}^{N} A_k^*(n)\left\{\left(1-\frac{k}{\mu}\right)e^{\mu/k} + \left(1+\frac{k}{\mu}\right)e^{-\mu/k}\right\} + R_2(n, N),$$

due respectively to Hardy and Ramanujan [1]‡ (1917) and to Rademacher [2] (1937). Here we have introduced the abbreviation

$$(3)\qquad \mu = \mu(n) = (\pi/6)(24n-1)^{1/2} = O(n^{1/2}).$$

The coefficients A^* are real numbers defined by

$$(4)\qquad A_k^*(n) = k^{-1/2}A_k(n),$$

where $A_k(n)$ is a complicated sum of $24k$th roots of unity.§ The remainders $R_1(n, N)$ and $R_2(n, N)$ are defined by (1) and (2) in which $p(n)$ denotes the number of unrestricted partitions of n.

The fact of primary importance about (2) is that

$$(5)\qquad \lim_{N\to\infty} R_2(n, N) = 0;$$

that is to say, the series in (2) as $N\to\infty$ converges for all n to $p(n)$. Concerning $R_1(n, N)$ Hardy and Ramanujan proved that for every $\alpha>0$

$$(6)\qquad R_1(n, \alpha n^{1/2}) = O(n^{-1/4}).$$

Rademacher [2] gave the following estimate for $R_2(n, N)$ in general:

$$(7)\qquad |R_2(n, N)| < \frac{44\pi^2}{225\cdot 3^{1/2}}N^{-1/2} + \frac{\pi\cdot 2^{1/2}}{75}\,\frac{N^{1/2}}{(n-1)^{1/2}}\sinh\frac{\pi(2n/3)^{1/2}}{N}$$

and a more complicated estimate for $R_1(n, N)$ from which (6) follows in case $N=\alpha n^{1/2}$. These estimates for the possible errors in (1) and (2) permitted for

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‡ The numbers in square brackets refer to the papers listed in the bibliography at the end of this paper.

§ For a complete definition of the A's see either [1, p. 85], [2, p. 242], or [3, pp. 271–273].

the first time the use of either (1) or (2) with absolute assurance. Using the estimate

$$(8)\qquad |A_k(n)| < 2k^{5/6}$$

instead of the trivial

$$(9)\qquad |A_k(n)| < k$$

previously employed, the writer obtained [4, 5]

$$(10)\qquad |R_2(n, N)| < \frac{\pi^2}{3^{1/2}} N^{-2/3}\left\{\left(\frac{N}{\mu}\right)^3 \sinh\frac{\mu}{N} + \frac{1}{6} - \left(\frac{N}{\mu}\right)^2\right\},$$

$$(11)\qquad \begin{aligned} |R_1(n, N)| < \frac{\pi^2 N^{7/3}}{3^{1/2}\mu^3}&\left\{\sinh\frac{\mu}{N} + \frac{1}{6}\left(\frac{\mu}{N}\right)^3\right. \\ &\left. + \left(1 + \frac{N}{\mu}\right)\left(\frac{1}{7} + \frac{1}{3}\mu^{1/3}N^{-5/3}\right)\right\}. \end{aligned}$$

If in (10) and (11) we substitute $N = \alpha n^{1/2}$, we find that in either case

$$(12)\qquad R_i(n, \alpha n^{1/2}) = O(n^{-1/3}) \qquad (i = 1, 2).$$

In §2 we show by a simple asymptotic argument that

$$(13)\qquad R_i(n, \alpha n^{1/2}) = O(n^{-1/2}\log n) \qquad (i = 1, 2),$$

a result, which in a sense, is the best possible. In §3 by a more precise treatment we obtain formulas similar to (10) and (11) but of which (13) rather than (12) is a special case.

Hardy and Ramanujan [1, p. 107] raised the question of the boundedness of $A_k(n)$ in discussing the possible convergence of (1) as $N\to\infty$. In proving the divergence of (1) the writer [6] employed a sequence of A's which, if they tended to zero, did not do so rapidly enough to render (1) convergent. Although this showed, in other words, that $R_1(n, N)$ tends to zero for no value of n, it did not remove the possibility of $R_1(n, N)$ ultimately oscillating between fixed limits. Incidentally to this discussion it was shown that $A_k(0)$ and $A_k(-1)$ are unbounded. Later [4, Theorem 11], it was proved that $A_k(n)$ is an unbounded function of k for infinitely many values of n. In §4 we show that this is true for every value of $n>0$ or $n<0$, proving in fact that for all n. $A_k(n) = \Omega(k^{1/2})$. (The interest in $A_k(n)$ is not confined to positive values of n [3, p. 83; 7, p. 466].) From this result it follows that $R_1(n, N)$ does not oscillate between fixed limits, the terms of the series in (1) being unbounded. It follows also that the kth term of (2) is greater in absolute value than $1/k^2$ for an infinity of k's despite the apparent rapidity of its convergence.

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2. **Proof of** (13). It is convenient to begin with

LEMMA 1. *If α is a positive constant, then for $s<1$,*

$$\sum_{k\leqq \alpha n^{1/2}} A_k^*(n)k^{-s} = O(n^{(1-s)/2}\log n), \tag{14}$$

and if $s>1$, then

$$\sum_{k>\alpha n^{1/2}} A_k^*(n)k^{-s} = O(n^{(1-s)/2}\log n). \tag{15}$$

Proof. By Theorem 8 of [4], $A_k^*(n)$ in absolute value does not exceed $2^{\omega(k)}$, the number of odd quadratfrei divisors of k, and hence does not surpass $\tau(k)$, the number of divisors of k. If therefore we denote, as usual, by $T(k)$ the sum function

$$T(k) = \sum_{\nu=1}^{k} \tau(\nu) = O(k\log k), \tag{16}$$

then we have

$$\begin{aligned}\sum_{a<k\leqq b} \left| A_k^*(n) \right| k^{-s} &= O\Big(\sum_{a<k\leqq b}\tau(k)k^{-s}\Big) = O\Big(\sum_{a<k\leqq b} T(k)\{k^{-s}-(k+1)^{-s}\}\Big)\\ &= O\Big(\sum_{a<k\leqq b} k(\log k)k^{-s-1}\Big) = O\Big(\sum_{a<k\leqq b} k^{-s}\log k\Big)\\ &= O\Big(\int_a^b x^{-s}\log x dx\Big) = O(b^{1-s}\log b - a^{1-s}\log a).\end{aligned}$$

To prove equation (14) set $a=1$ and $b=\alpha n^{1/2}$. To prove (15) set $a=\alpha n^{1/2}$, $b=\infty$.

THEOREM 1. $R_2(n, \alpha n^{1/2}) = O(n^{-1/2}\log n)$.

Proof. If we expand the exponentials in (2) and collect the terms, we have

$$R_2(n, N) = \frac{4\cdot 12^{1/2}}{24n-1}\sum_{k=N+1}^{\infty} A_k^*(n)\sum_{j=1}^{\infty}\frac{j(\mu/k)^{2j}}{(2j+1)!}\,. \tag{17}$$

Hence in view of (3)

$$R_2(n, \alpha n^{1/2}) = O\Big(\frac{1}{n}\sum_{j=1}^{\infty}\frac{jn^j}{(2j+1)!}\sum_{k>\alpha n^{1/2}} \left| A_k^*(n)\right| k^{-2j}\Big).$$

Applying (15) with $s=2j$, we have

$$R_2(n, \alpha n^{1/2}) = O\left(\frac{1}{n} \sum_{j=1}^{\infty} \frac{jn^j}{(2j+1)!} n^{(1-2j)/2} \log n\right) = O(n^{-1/2} \log n).$$

It remains to prove

THEOREM 2. $R_1(n, \alpha n^{1/2}) = O(n^{-1/2} \log n)$.

Proof. Let $D(n, N)$ represent the difference between the sum of the first N terms of (1) and the first N terms of (2). Then in view of Theorem 1 it suffices to show that $D(n, \alpha n^{1/2}) = O(n^{-1/2} \log n)$. Now

$$D(n, \alpha n^{1/2}) = \frac{12^{1/2}}{24n - 1} \sum_{k \leqq \alpha n^{1/2}} A_k^*(n) \left(1 + \frac{k}{\mu}\right) e^{-\mu/k}.$$

Since $e^{-\mu/k} < k/\mu$, we have by (3)

$$D(n, \alpha n^{1/2}) = O\left(\frac{1}{n}\left\{ \sum_{k \leqq \alpha n^{1/2}} | A_k^*(n) | \frac{k}{n^{1/2}} + \sum_{k \leqq \alpha n^{1/2}} | A_k^*(n) | \frac{k^2}{n}\right\}\right).$$

Applying (14) with $s = -1$ and $s = -2$, we have

$$D(n, \alpha n^{1/2}) = O(n^{-3/2} n \log n + n^{-2} n^{3/2} \log n) = O(n^{-1/2} \log n).$$

This completes the proof of (13).

3. **Estimates of the general remainders** $R_i(n,N)$. In what follows we shall use the function $T(n)$ as before but will require something more explicit than (16). Hence we start with

LEMMA 2. *For all positive integers* k

$$T(k) > k \log k, \tag{18}$$

while for $k > 12$

$$T(k) \leqq k(\log k + b_{16}) \tag{19}$$

where $b_{16} = T(16)/16 - \log 16 = .3524113 \cdots$.

Proof. We shall need the following inequalities:

$$\log x - \delta/(x - \delta) < \log (x - \delta) < \log x - \delta/x, \qquad 0 < \delta < 1 < x, \tag{20}$$

$$\log m + C < H(m) < \log m + C + 1/(2m) \tag{21}$$

where $H(m) = 1 + 1/2 + 1/3 + \cdots + 1/m$ and $C = .577215 \cdots$ is Euler's constant. The inequalities (20) follow readily from $e^{\delta/(x-\delta)} > 1/(1 - \delta/x) > e^{\delta/x}$, which are seen at once to be true on expanding the functions involved, while (21) follows from the familiar asymptotic expansion

$$H(m) = \log m + C + 1/(2m) - 1/(12m^2) - \cdots.$$

If in the well known relation

$$T(k) = 2 \sum_{x \leqq k^{1/2}} \left[\frac{k}{x}\right] - [k^{1/2}]^2 \tag{22}$$

we remove the greatest integer signs in the sum, we obtain by (21)

$$T(k) \leqq 2kH([k^{1/2}]) - [k^{1/2}]^2 < 2k \log [k^{1/2}] + 2Ck + k/[k^{1/2}] - [k^{1/2}]^2. \tag{23}$$

Writing

$$[k^{1/2}] = k^{1/2} - \delta \tag{24}$$

and applying (20) we have at once from (23)

$$\begin{aligned} T(k) &< k \log k + (2C - 1)k + k/(k^{1/2} - \delta) - \delta^2 \\ &< k(\log k + 2C - 1 + 1/(k^{1/2} - 1)). \end{aligned}$$

Now if $k \geqq 37$, then

$$2C - 1 + 1/(k^{1/2} - 1) < .15444 + 1/(37^{1/2} - 1) < .35122 < b_{16}.$$

Hence (19) is true for $k > 36$. That it is true for $12 < k \leqq 36$ is shown by the following table of $T(k)$ and

$$b_k = T(k)/k - \log k \tag{25}$$

which will be of use later.

TABLE I

k	$T(k)$	b_k	k	$T(k)$	b_k	k	$T(k)$	b_k	k	$T(k)$	b_k
1	1	1.000	11	29	.238	21	70	.289	31	113	.211
2	3	.807	12	35	.432	22	74	.273	32	119	.253
3	5	.568	13	37	.238	23	76	.169	33	123	.231
4	8	.614	14	41	.288	24	84	.322	34	127	.209
5	10	.391	15	45	.292	25	87	.261	35	131	.188
6	14	.541	16	50	.352	26	91	.242	36	140	.305
7	16	.340	17	52	.225	27	95	.186	37	142	.227
8	20	.421	18	58	.332	28	101	.275	38	146	.168
9	23	.358	19	60	.213	29	103	.184	39	150	.183
10	27	.397	20	66	.304	30	111	.299	40	158	.261

It is seen that (19) is true also for $k=7$ and 11 and that the equal sign holds only when $k=16$. Any number of inequalities similar to (19), such as

$$T(k) < k(\log k + b_{18}) < k(\log k + .33185047), \qquad k > 16,$$

may be established in the same way.

To prove (18) we use the inequality

$$[k/x] \geqq (k - x + 1)/x,$$

so that (22) gives

$$T(k) \geqq 2 \sum_{x<[k^{1/2}]} \frac{k+1}{x} - 2[k^{1/2}] - [k^{1/2}]^2$$
$$= 2(k+1)H([k^{1/2}]) - 2[k^{1/2}] - [k^{1/2}]^2.$$

By (20), (21) and (24) we have therefore

$$\begin{aligned} T(k) &> (k+1)(\log k - 2\delta/(k^{1/2}-\delta) + 2C) - (1+[k^{1/2}])^2 + 1 \\ &> (k+1)\log k + (2C-1)k + 2C - 2k^{1/2} - 2(k^{1/2}+1)/(k^{1/2}-1) \\ &= k\log k + (2C-1)k + \log k - 2((k+1)/(k^{1/2}-1) - C). \end{aligned}$$

The function $(2C-1)k+\log k-2((k+1)/(k^{1/2}-1)-C)>0$ for $k>117$. Hence (18) is true, if true for $k\leqq 117$, and this is readily verified. In fact in view of (25) it is seen that (18) is equivalent to $b_k>0$, an inequality which holds for $k\leqq 117$, the smallest value of b_k being $b_{59}=.14280154$. Of course b_k tends to $2C-1=.1544\cdots$ as $k\to\infty$.

LEMMA 3. *Let $N>12$, $s>1$, and let n be any integer. Then*

$$\sum_{k=N}^{\infty} |A_k^*(n)|\, k^{-s} < -T(N-1)N^{-s} + s(s-1)^{-1}N^{1-s}\{\log N + (s-1)^{-1} + .3524113\}. \tag{26}$$

Proof. From the fact that

$$|A_k^*(n)| \leqq \tau(k) \tag{27}$$

it follows that

$$\begin{aligned} \sum_{k=N}^{\infty} |A_k^*(n)|\, k^{-s} \leqq \sum_{k=N}^{\infty} \tau(k)k^{-s} &= \sum_{k=N}^{\infty} T(k)\{k^{-s} - (k+1)^{-s}\} - T(N-1)N^{-s} \\ &< -T(N-1)N^{-s} - \int_N^{\infty} T(x)d(x^{-s}) \\ &= -T(N-1)N^{-s} + s\int_N^{\infty} T(x)x^{-s-1}dx. \end{aligned}$$

Here we have defined $T(x)$ as $T([x])$. Applying Lemma 1 with $k=N>12$ and integrating, we obtain the lemma at once.

THEOREM 3. *If $N>12$, then*

$$R_2(n, N-1) < (48^{1/2}\pi^2/9N)\{w_1(\mu/N)(\log N + .3524113) + w_2(\mu/N) - (T(N-1)/2N)w_3(\mu/N)\}$$

where the functions $w_i(x)$ are defined by

$$(28)\qquad w_1(x) = \sum_{j=1}^{\infty} \frac{j^2 x^{2j-2}}{(2j-1)(2j+1)!}, \qquad w_2(x) = \sum_{j=1}^{\infty} \frac{j^2 x^{2j-2}}{(2j-1)^2(2j+1)!},$$
$$w_3(x) = \sum_{j=1}^{\infty} \frac{j x^{2j-2}}{(2j+1)!}.$$

Proof. By (17) we have

$$R_2(n, N-1) = \frac{4\cdot 12^{1/2}}{24n-1} \sum_{j=1}^{\infty} \frac{j\mu^{2j}}{(2j+1)!} \sum_{k=N}^{\infty} A_k^*(n) k^{-2j}.$$

Taking absolute values and applying Lemma 3 with $s=2j$, we find that

$$| R_2(n, N-1) | < \frac{4\cdot 12^{1/2}\mu^2}{(24n-1)N} \left\{ -\frac{T(N-1)}{N} \sum_{j=1}^{\infty} \frac{j(\mu/N)^{2j-2}}{(2j+1)!} \right.$$
$$\left. + 2(\log N + b_{16}) \sum_{j=1}^{\infty} \frac{j^2(\mu/N)^{2j-2}}{(2j-1)(2j+1)!} + 2\sum_{j=1}^{\infty} \frac{j^2(\mu/N)^{2j-2}}{(2j-1)^2(2j+1)!} \right\}.$$

Making use of (3) and (28), we obtain the theorem at once.

THEOREM 4. *If N is any positive integer,*

$$| R_1(n, N) | < (72\cdot 3^{1/2}/\pi^2)(N/\mu)^2 e^{-\mu/N}\{(T(N)/N^2)(1 + (N+1)/\mu)$$
$$- (1/2\mu)(\log N - 1/2)\} + | R_2(n, N) |.$$

Proof. As before let $D(n, N)$ denote the difference between the sums of the first N terms of (1) and (2) so that

$$D(n, N) = \frac{12^{1/2}}{24n-1} \sum_{k=1}^{N} A_k^*(n) \left(1 + \frac{k}{\mu}\right) e^{-\mu/k}.$$

Then

$$(29)\qquad | R_1(n, N) | < | R_2(n, N) | + | D(n, N) |.$$

Now

$$(30)\qquad | D(n, N) | < \frac{12^{1/2}e^{-\mu/N}}{24n-1} \left\{ \sum_{k=1}^{N} | A_k^*(n) | + \frac{1}{\mu} \sum_{k=1}^{N} k\, | A_k^*(n) | \right\}$$

but from (27)

$$\sum_{k=1}^{N} | A_k^*(n) | \leqq T(N)$$

while

$$(31)\qquad \sum_{k=1}^{N} k\, | A_k^*(n) | \leqq \sum_{k=1}^{N} k\tau(k) = (N+1)T(N) - \sum_{k=1}^{N} T(k).$$

By (18) we have

$$\sum_{k=1}^{N} T(k) > \sum_{k=1}^{N} k \log k > \int_1^N x \log x dx > \frac{1}{2} N^2 \log N - \frac{N^2}{4},$$

so that by (30) and (31)

$$|D(n, N)| < \frac{12^{1/2}e^{-\mu/N}}{24n - 1}\left\{T(N) + \frac{N+1}{\mu} T(N) - \frac{N^2}{4\mu}(2 \log N - 1)\right\}.$$

The theorem now follows from (3) and (29).

Of the three functions (28) only $w_3(x)$ is elementary; in fact

$$w_3(x) = \frac{1}{2x^2}(x \cosh x - \sinh x),$$

the other two depending on higher transcendents. For our purposes it is best to use their series not only as definitions but also as effective means of evaluating these functions A short table of $w_1(x)$, $w_2(x)$ and $w_3(x)$ is given below.

TABLE II

x	$w_1(x)$	$w_2(x)$	$w_3(x)$
1	.1781	.1704	.1839
2	.2172	.1827	.2436
3	.3007	.2065	.3738
4	.4668	.2485	.6402
5	.7992	.3215	1.1874
6	1.4794	.4535	2.3347
7	2.9089	.6873	4.7958
8	5.9877	1.1327	10.1901
9	12.7652	1.9991	22.2307
10	27.9660	3.7336	49.5596

In actual practice we are concerned with $n>600$, since tables of $p(n)$ now extend to $p(600)$. Unless we carry the calculation to a considerable number of places to the right of the decimal point and at the same time employ quite a large number of terms, we cannot distinguish between the terms of (1) and (2). Hence in practice we may use (1) and apply Theorem 3 to estimate the remainder. We give three examples of the application of above estimates

TABLE III

	By (7)	*By* (10)	*Theorem* 3	*Actual value*	μ
$\lvert R_2(721,21)\rvert$	.378	.341	.231	.00041	68.875
$\lvert R_2(2052,18)\rvert$	2.028	1.099	.815	.0408	116.20
$\lvert R_2(14031,63)\rvert$	.387	.245	.150	.00016	303.84

in widely different cases. Linear interpolation may be used for $w_1(x)$ and $w_2(x)$, since it will give values in excess of the actual values of these functions.

Values of $T(n)$ can be taken from Table I for $n \leqq 40$ and can be quickly found from (22) if $n > 40$. For rough calculation we may use the inequality

$$T(N-1)/2N > \log N^{1/2}.$$

If we replace N by $\alpha n^{1/2}$ so that

$$\mu/N = O(1), \qquad T(N)/N = O(n^{-1/2} \log n)$$

in Theorems 3 and 4, it is seen that (13) is a special case of these theorems.

If instead of (27) we were to use

$$|A_k^*(n)| \leqq 2^{\omega(k)}, \tag{32}$$

then for very large values of N it would be possible to obtain smaller estimates for $|R_i(n, N)|$ by a simple modification of the above argument. In fact we would then be concerned with the function

$$\psi(k) = \sum_{\nu=1}^{k} 2^{\omega(\nu)} = \frac{3}{\pi^2} k \log k + O(k), \tag{33}$$

so that theoretically one could reduce the estimate for $|R_i(n, N)|$ by a factor of nearly $3/\pi^2$. This of course would not alter (13). If one is to use some inequality for $|A_k^*(n)|$ of the same type as (27) or (32) in which the right side is independent of n, then it is impossible to obtain an essentially better inequality than (32). In this sense (13) cannot be improved upon. In practice a small bound for the constant implied in (33) is not easily obtained, nor indeed can one achieve the factor† $3/\pi^2$. In the end one obtains theorems similar to Theorems 3 and 4 which are superior only for larger values of N than one would naturally encounter in actual calculations.

4. **Proof of unboundedness of $A_k(n)$.** In proving that $A_k(n)$ is unbounded it is necessary to consider separately the cases in which n is and is not the negative of a pentagonal number. In the first case the proof is quite simple. In the second case we make use of a lemma depending on the prime number theorem.

THEOREM 5. *If $-n$ is a pentagonal number, there exist infinitely many primes p such that $|A_p(n)| > (3p)^{1/2}$. This is not true for a larger number than* 3.

Proof. By Theorem 5 of [4]

$$|A_p(n)| = 2p^{1/2} |\cos(4\pi m/p)| \tag{34}$$

where the integer m satisfies the congruence

$$(24m)^2 \equiv 1 - 24n \not\equiv 0 \pmod{p}. \tag{35}$$

† One can prove for instance that $\psi(k) < .6534k \log k + 3.387k$.

If $-n$ is a pentagonal number so that $n=-(3u^2\pm u)/2$ or rather $1-24n=(6u\pm 1)^2$, the congruence (35) has for every prime $p>3$ not dividing $1-24n$ the integral solution $m=(6u\pm 1)(p^2-1)/24$. Hence by (34)

$$\begin{aligned} |A_p(n)| &= 2p^{1/2}\left|\cos\left\{(6u\pm 1)(p-1/p)\pi/6\right\}\right| \\ &= 2p^{1/2}\left|\cos\left\{(p-[6u\pm 1]/p)\pi/6\right\}\right|. \end{aligned}$$

As $p\to\infty$ through prime values this tends steadily to $2p^{1/2}\cos\pi/6=(3p)^{1/2}$ and approaches it from above or below according as $p\to\infty$ through values of the form $6k+1$ or $6k-1$. Thus the assertions of the theorem are proved.

LEMMA 4. *Let a and b be coprime integers such that $-ab$ is a non-square and a is even, and let t_M denote the least positive solution (if it exists) of the congruence*

$$at^2+b\equiv 0 \pmod{M}.$$

Finally let γ be a constant greater than $1/2$; *then the inequality*

$$t_p<\gamma p/\log p$$

holds for infinitely many primes p.

Proof. Let $\sum'$ denote a summation over those primes greater than 2 of which $-ab$ is a quadratic residue. We recall that asymptotically half the primes are of this sort. Let x be a large integer. Then identically

$$\sum_{k=1}^{x}\log|ak^2+b| = \sum{}'(\log p)\left\{\left[\frac{x+p-t_p}{p}\right]+\left[\frac{x+t_p}{p}\right] + \sum_{\nu=2}^{\infty}\left(\left[\frac{x+p^\nu-t_{p^\nu}}{p^\nu}\right]+\left[\frac{x+t_{p^\nu}}{p^\nu}\right]\right)\right\}. \tag{36}$$

Now

$$\sum_{k=1}^{x}\log|ak^2+b| = 2x\log x+O(x).$$

The right member of (36) may be written

$$\begin{aligned} \sum_{p<2x}{}'\left(\frac{2x}{p}+1\right)\log p + O\left(\sum_{p<2x}{}'\log p\right) + \sum_{p^\nu<ax^2+b,\,\nu>1}{}'\left(\frac{2x}{p^\nu}+1\right)\log p \\ + \sum_{p>2x,\,t_p<x}{}'\log p+O(x) = 2x\sum_{p<2x}{}'\log p/p + \sum_{p>2x,\,t_p<x}{}'\log p+O(x) \\ = x\log x+\sum_{p>2x,\,t_p<x}\log p+O(x). \end{aligned}$$

Now suppose that the lemma is false so that $t_p\geqq\gamma p/\log p$ for all sufficiently large p. Then for a sufficiently large p the inequality $t_p<x$ implies $x>\gamma p/\log p$. That is

$$p < (1/\gamma)x \log p = (1/\gamma)x \log x + O(x \log \log x).$$

Hence

$$\begin{aligned}\sum_{p>2x,\, t_p<x}{}' \log p &= \sum_{p>2x,\, p<x(\log x)/\gamma}{}' \log p + O(x \log \log x)\\ &= (x/2\gamma) \log x + O(x \log \log x).\end{aligned}$$

Therefore (36) may be written for all sufficiently large x

$$2x \log x = x \log x + (1/2\gamma)x \log x + O(x \log \log x)$$

or

$$(2\gamma - 1)x \log x = O(x \log \log x).$$

But as $\gamma>1/2$, this is a contradiction. Hence the lemma is true.

The proof of this lemma was suggested to the writer by Dr. H. Heilbronn. As a matter of fact the hypotheses of the lemma are unnecessarily restrictive but are sufficient to meet our immediate needs. By only a slight complication of (36) the same proof applies to any irreducible quadratic.

THEOREM 6. *If $-n$ is not a pentagonal number, there exist for every $\epsilon>0$ infinitely many primes p such that*

$$|A_p(n)| > (2 - \epsilon)p^{1/2}. \tag{37}$$

Proof. In Lemma 4 choose $a=24^2$ and $b=24n-1$. This is permissible since a is even and prime to b, and $-ab=24^2(1-24n)$ is a non-square because $-n$ is not a pentagonal number. Then by Lemma 4 there exist infinitely many primes p for which the congruence (35) has a solution m such that $0<m<\eta p$ where η is a positive constant less than 1/8 to be determined presently. Then for every such p, by (34),

$$|A_p(n)| > 2p^{1/2} \cos 4\pi\eta.$$

To obtain (37) one has only to choose η so small that $\cos 4\pi\eta$ differs from unity by less than $\epsilon/2$.

THEOREM 7. *For every positive n the kth term of the Hardy-Ramanujan series* (1) *is in absolute value greater than*

$$13k(24n - 1)^{-3/2}$$

for infinitely many values of k.

Proof. By Theorem 6 for every $\epsilon>0$ there exist infinitely many primes p

for which the pth term of (1) is greater in absolute value than

$$\frac{12^{1/2}}{24n-1}\left|1-\frac{p}{\mu}\right|e^{\mu/p}(2-\epsilon) > \frac{6\cdot 12^{1/2}p(2-\epsilon)}{\pi(24n-1)^{3/2}} - \frac{2e12^{1/2}}{24n-1}$$

provided $p>\mu$. There exists a positive ϵ so small that $6\cdot 12^{1/2}(2-\epsilon)/\pi>13$. For such an ϵ and for all sufficiently large primes p associated with this ϵ, the pth term of (1) is greater than $13p(24n-1)^{-3/2}$.

Theorem 8. *For every positive n the kth term of the Rademacher series* (2) *is in absolute value greater than* $(43/34)k^{-2}$ *for infinitely many k's.*

Proof. Since

$$(1-x^{-1})e^{x}+(1+x^{-1})e^{-x} = 2(x^2/3+x^4/30+\cdots) > (2/3)x^2,$$

the pth term of (2) in view of (3), (4) and (37) is, in absolute value, greater than

$$\frac{12^{1/2}}{24n-1}\cdot\frac{2}{3}\,\frac{\mu^2}{p^2}(2-\epsilon) = \frac{2\pi^2}{9\cdot 3^{1/2}}\,p^{-2} - \epsilon' p^{-2} > \frac{43}{34}\,p^{-2}$$

for a suitably chosen ϵ.

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